Southern University of Science and Technology

Berry-Essen Bounds of Nonnormal Approximation for Bounded and Unbounded Exchangeable Pairs [Shao and Zhang, 2019]

Hongxia Jin
Yixuan Liu

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Exchangeable Pairs

Definition 1.1

(W, W') is said to be an exchangeable pair if

$$(W, W') \stackrel{\text{d.}}{=} (W', W)$$

for which (W, W') and (W', W) have the same joint distribution.

A key fact about an exchangeable pair (W, W') is that for any asymmetric $h(x, y)$, that is $h(x, y) = -h(y, x)$, we have

$$\mathrm{E}h(W, W') = 0$$

by noting that $\mathrm{E}h(W, W') = \mathrm{E}h(W, W') = -\mathrm{E}h(W, W')$.

Exchangeable Pairs

When we assume that (W, W') is an exchangeable pair and let $\Delta = W - W'$, another key fact is that for any f,

$$E(f(W)E(\Delta \mid W)) = \frac{1}{2}E\int_{-\infty}^{\infty} f'(W+t)\Delta\left(\mathbb{1}_{\{-\Delta\leqslant t<0\}} - \mathbb{1}_{\{0\leqslant t<-\Delta\}}\right)dt \tag{1}$$

Proof. Consider $h(w, w') = (w - w')(f(w) + f(w'))$, which is an asymmetric function. Then,

$$\begin{aligned} 0 &= \mathrm{E}h(W, W') = \mathrm{E}\left[(W - W')(f(W) + f(W'))\right] \\ &= 2\mathrm{E}[\Delta f(W)] - \mathrm{E}[\Delta(f(W) - f(W-\Delta))] \\ &= 2\mathrm{E}[f(W)E(\Delta \mid W)] - \mathrm{E}\left[\Delta\int_{-\Delta}^{0} f'(W+t)dt\right] \\ &= 2\mathrm{E}[f(W)E(\Delta \mid W)] - \mathrm{E}\left[\int_{-\infty}^{\infty} f'(W+t)\Delta\left(\mathbb{1}_{\{-\Delta\leqslant t<0\}} - \mathbb{1}_{\{0\leqslant t<-\Delta\}}\right)dt\right] \end{aligned}$$

The Stein Characterization via Density Approach

Let Y be a random variable with the probability density function $p(y)$. Assume that $p(y) > 0$ for $\infty < y < \infty$. Let f be an absolutely continuous function satisfying $lim_{y \Rightarrow \pm\infty} p(y)f(y) = 0$, we have

$$\mathrm{E}\left\{(f(Y)p(Y))'/p(Y)\right\} = \int_{-\infty}^{\infty} (f(y)p(y))' dy = 0$$

Stein's identity:

$$\mathrm{E}(f(Y)p(Y))'/p(Y)) = 0$$

For any measurable function h with $\mathrm{E}\|h(Y\| < \infty$, let $f = f_h$ be the solution to the Stein equation:

$$(f(w)p(w))'/p(w) = h(w) - \mathrm{E}h(Y) \tag{2}$$

Nonnormal Approximation with Exchangeable Pairs

In this section, we are going to review the Berry-Essen bounds for non-normal approximation with both bounded and unbounded exchangeable pairs.

Let W be a random variable satisfying $P(a < W < b) = 1$ where $-\infty \leq a \leq \infty$. Let (W, W') be an exchangeable pair satisfying

$$\mathrm{E}(W - W' \mid W) = \lambda(g(W) + R) \tag{3}$$

Nonnormal Approximation for Unbounded Exchangeable Pairs

Consider when g is a measurable function with domain (a, b), $\lambda \in (0, 1)$ and R is a random variable.

Conditions of measurable function g

- A1 g is nondecreasing, and there exists $w_0 \in (a, b)$ such that $(w - w_0)g(w) \geq 0$ for $w \in (a, b)$;
- A2 g' is continuous and $2(g'(w))^2 - g(w)g''(w) \geq 0$ for all $w \in (a, b)$; and
- A3 $\lim_{y \downarrow a} g(y)p(y) = \lim_{y \uparrow b} g(y)p(y) = 0$, where

$$p(y) = c_1 e^{-G(y)}, \quad G(y) = \int_{w_0}^{y} g(t)dt \tag{4}$$

and c_1 is the constant so that $\int_a^b p(y)dy = 1$.

Berry-Essen Bound with Unbounded Exchangeable Pairs

Let Y be a random variable with the probability density function (p.d.f.) $p(y)$, and let $\Delta = W - W'$.

Theorem 2.1

We have

$$\sup_{z\in\mathbb{R}} |\mathrm{P}(W \le z) - \mathrm{P}(Y \le z)| \le \mathrm{E}\left|1 - \frac{1}{2\lambda}\mathrm{E}\left(\Delta^2 \mid W\right)\right| + \frac{1}{\lambda}\mathrm{E}\left|\mathrm{E}\left(\Delta\Delta^* \mid W\right)\right| + \frac{1}{c_1}\mathrm{E}|R|, \tag{5}$$

where $\Delta^* := \Delta^*(W, W')$ is any random variable satisfying $\Delta^*(W, W') = \Delta^*(W', W)$ and $\Delta^* \ge |\Delta|$.

To make the bound meaningful, choose $\lambda \sim (1/2)\mathrm{E}(\Delta^2)$.

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Stein Equation and Solution Properties

Let Y be the random variable with the p.d.f. $p(y)$ defined in (4). For a given z, let $f := f_z$ be the solution to the following Stein equation:

$$f'(w) - g(w)f(w) = \mathbb{1}_{\{w \leq z\}} - F(z), \quad z \in (a, b) \tag{6}$$

where F is the distribution function of Y . It could be derived from (2), $(f(w)p(w))'/p(w) = h(w) - \mathrm{E}h(Y)$ with $h(w) = \mathbb{1}_{\{w \leq z\}}$.

$$f_z(w) = \begin{cases} \frac{F(w)(1-F(z))}{p(w)}, & w \leq z, \\ \frac{F(z)(1-F(w))}{p(w)}, & w > z \end{cases} \tag{7}$$

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Stein Equation and Solution Properties

We first prove some basic properties of f_z.

Lemma 2.3 (Basic properties of f)

Suppose that conditions (A1)-(A3) are satisfied. Then

$$0 \le f_z(w) \le 1/c_1, \tag{8}$$

$$\|f'\| \le 1, \tag{9}$$

$$\|gf_z\| \le 1 \tag{10}$$

$$g(w)f_z(w) \text{ is nondecreasing} \tag{11}$$

For the normal approximation, it is known that $0 \le f_z(w) \le 1$.

Proof of Properties

Prove (8), $0 \le f_z(w) \le 1/c_1$**:**

Assume $a < 0 < b$ and $w_0 = 0$; thus $p(0) = c_1$. For $w \le z$, define $\boldsymbol{H_z(w) = F(w)(1 - F(z)) - p(w)/c_1}$. Noting that $f_z(w) \ge 0$ according to (7). It suffices to prove $sup_{a<w<b} H_z(w) \le 0$. Since $g(w)$ is nondecreasing and $H_z' = p(w)(1 - F(z) + g(w)/c_1)$ $(-g(w) = \frac{p'(w)}{p(w)})$, maximum only occurs at boundaries.

$$\sup_{a<w\le z} H_z(w) = \max\{H_z(a), H_z(z)\}.$$

$H_z(a) = -p(a)/c_1 \le 0$. It remains to prove $sup_{a\le z\le b} H_z(z) \le 0$. If $z \le 0$, define $H_1(z) = F(z) - p(z)/c_1$, and thus

$$H_1'(z) = p(z)(1 + g(z)/c_1).$$

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Note that $g(z) \leq 0$ and $g(\cdot)$ is nondecreasing, $H_1(a) = -p(a)/c_1$, $H_1(0) = F(z) - 1$, then

$$\sup_{a<z\leq 0} H_z(z) \leq \sup_{a<z\leq 0} H_1(z) \leq \max\{H_1(a), H_1(0)\} \leq 0$$

Using a similar argument, we also have $sup_{0\leq z\leq b} H_z(z) \leq 0$. Therefore, $sup_{a<z<b} H_z(z) \leq 0$. $sup_{a\leq w\leq z} f_z(w) \leq 1/c_1$ is proved. Similarly, we have $sup_{z<w<b} f_z(w) \leq 1/c_1$. Similar procedures could be made for $z \geq 0$ when $sup_{0\leq z<b}$, and complete the proof of $sup_{a\leq w\leq z} f_z(2) \leq 1/c_1$ with $w > z$. Similarly for $w > z$, $sup_{z<w<b} f_z(w) \leq 1/c_1$.

Prove (11), $g(w)f_z(w)$ is nondecreasing:

gf_z is nondecreasing. For $w \le z$, by (7),

$$g(w)f_z(w) = \frac{g(w)F(w)(1-F(z))}{p(w)}$$

$$\begin{aligned}(g(w)f_z(w))' &= (1-F(z))\left(g(w) + \left(g'(w)+g^2(w)\right)F(w)/p(w)\right)\\ &= \underbrace{\frac{c_1(1-F(z))(g'(w)+g^2(w))}{p(w)}}_{\ge 0}\underbrace{(\tau(w)+\frac{F(w)}{c_1})}_{WTP\ge 0}\end{aligned}$$

If $\tau(w) = \frac{g(w)e^{-G(w)}}{g'(w)+g^2(w)}$, by (A2) ($2(g'(w))^2 - g(w)g''(p(w)) \ge 0$),

$$-\tau'(w)e^{G(w)} = 1 - \left(\frac{2\left(g'(w)\right)^2 - g''(w)g(w)}{\left(g'(w)+g^2(w)\right)^2}\right) \le 1.$$

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Hence,

$$e^{-G(w)} + \tau'(w) \geq 0$$

and

$$0 \leq \int_a^w \left(\tau'(t) + e^{-G(t)}\right) dt = \tau(w) + \frac{1}{c_1} F(w) - \lim_{y \downarrow a} \tau(y)$$

Where

$$\int_a^w e^{-G(t)} dt = \int_w^a \frac{p(y)}{c_1} = \frac{1}{c_1}(F(w) - F(a)) = \frac{F(w)}{c_1}$$

By condition (A3), $\lim_{y \downarrow a} \tau(y) = 0$, and hence $\tau(w) + \frac{1}{c_1} F(w) \geq 0$. This proves that $(g(w) f_z(w))' \geq 0$ or $g(w) f_z(w)$ is nondecreasing for wz. Similarly for $w \geq z$.

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Prove (10), $\|gf_z\| \le 1$:
we have for $w \ge \max(z, 0)$,

$$g(w)f_z(w) = \frac{F(z)g(w)\int_w^b p(t)dt}{p(w)} \le \frac{F(z)\int_w^b g(t)p(t)dt}{p(w)}$$

$$\overset{p(w)=c_1e^{-G(w)}}{=} \frac{F(z)\int_w^b e^{-G(t)}g(t)dt}{e^{-G(w)}} = F(z)\left(\frac{e^{-G(w)} - e^{-G(b)}}{e^{-G(w)}}\right)$$

$$\le F(z).$$

Similarly, we have $g(w)f_z(w) \ge -(1 - F(z))$ for $w \le \min(0, z)$.
Combining with (11) yields

$$F(z) - 1 \le g(w)f_z(w) \le F(z) \tag{12}$$

for all w.

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Prove (9), $\|f'\| \le 1$: Prove of (9) follows from (6) and (12). Where $\mathbb{1}_{\{w \le z\}} - 1 \le f'(w) \le \mathbb{1}_{\{w \le z\}}$. □

Proof of Theorem 2.1

Let $f = f_z$ be the solution to the Stein equation (7). (W, W') is an exchangeable pair, by (3), $E(W - W' \mid W) = \lambda(g(W) + R)$, we have:

$$
\begin{aligned}
0 &= \mathrm{E}\left(\left(W - W'\right)\left(f(W) + f\left(W'\right)\right)\right) \\
&= 2\mathrm{E}\left(\left(W - W'\right) f(W)\right) - \mathrm{E}\left(\left(W - W'\right)\left(f(W) - f\left(W'\right)\right)\right) \\
&= 2\lambda \mathrm{E}(g(W) f(W)) + 2\lambda \mathrm{E}(R f(W)) - \mathrm{E}\left(\Delta \int_{-\Delta}^{0} f'(W + t) dt\right),
\end{aligned}
$$

and hence

$$
\mathrm{E}(g(W) f(W)) = \frac{1}{2\lambda} \mathrm{E}\left(\Delta \int_{-\Delta}^{0} f'(W + t) dt\right) - \mathrm{E}(R f(W)).
$$

Thus,

$$\begin{aligned}
&\mathrm{E}\left(f'(W)-g(W)f(W)\right)\\
&=\mathrm{E}\left(f'(w)-\frac{1}{2\lambda}\Delta\int_{-\Delta}^{0}(f'(w+t)-f'(w)+f'(w))dt+Rf(w)\right)\\
&=\mathrm{E}\left(f'(W)\left(1-\frac{1}{2\lambda}\mathrm{E}\left(\Delta^2\mid W\right)\right)\right)\\
&\underbrace{-\frac{1}{2\lambda}\mathrm{E}\left(\Delta\int_{-\Delta}^{0}\left(f'(W+t)-f'(W)\right)dt\right)}_{I_1}+\mathrm{E}(Rf(W))
\end{aligned}$$

By (8) , (9) and (10) and using Stein equation,

$$\begin{aligned}
&|\mathrm{P}(W\le z)-\mathrm{P}(Y\le z)|=|\mathrm{E}\left(f'(W)-g(W)f(W)\right)|\\
&\le|I_1|+2\mathrm{E}\left|1-\frac{1}{2\lambda}\mathrm{E}\left(\Delta^2\mid W\right)\right|+\frac{1}{c_1}\mathrm{E}|R|,
\end{aligned}\tag{13}$$

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where

$$I_1 = \frac{1}{2\lambda}\mathrm{E}\left(\Delta \int_{-\Delta}^{0} \left(f'(W+t) - f'(W)\right) dt\right) \tag{14}$$

Recalling that f is the solution to the Stein equation (3), we have

$$\begin{aligned} I_1 =& \frac{1}{2\lambda}\mathrm{E}\left(\Delta \int_{-\Delta}^{0} (g(W+t)f(W+t) - g(W)f(W))dt\right) \\ &+ \frac{1}{2\lambda}\mathrm{E}\left(\Delta \int_{-\Delta}^{0} \left(\mathbb{1}_{\{W+t\le z\}} - \mathbb{1}_{\{W\le z\}}\right) dt\right). \end{aligned} \tag{15}$$

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Noting that $g(w)f(w)$ is nondecreasing by Lemma 2.3 and that the indicator function $\mathbb{1}_{\{w\le z\}}$ is nonincreasing, we have

$$\begin{aligned} 0 &\ge \int_{-\Delta}^{0} (g(W+t)f(W+t) - g(W)f(W))dt \\ &\ge -\Delta(g(W)f(W) - g(W-\Delta)f(W-\Delta)) \end{aligned}$$

and

$$0 \le \int_{-\Delta}^{0} \left(\mathbb{1}_{\{W+t\le z\}} - \mathbb{1}_{\{W\le z\}}\right) dt \le \Delta\left(\mathbb{1}_{\{W-\Delta\le z\}} - \mathbb{1}_{\{W\le z\}}\right)$$

Therefore

$$\begin{aligned} I_1 \le& \frac{1}{2\lambda} E\left(-\Delta \mathbb{1}_{\{\Delta<0\}} \Delta(g(W)f(W) - g(W-\Delta)f(W-\Delta))\right) \\ &+ \frac{1}{2\lambda} E\left(\Delta \mathbb{1}_{\{\Delta>0\}} \Delta\left(\mathbb{1}_{\{W-\Delta\le z\}} - \mathbb{1}_{\{W\le z\}}\right)\right) \end{aligned} \tag{16}$$

Thus, for any $\Delta^* = \Delta^*(W, W') = \Delta^*(W', W) \geq |\Delta|$.

$$\begin{aligned}
&\frac{1}{2\lambda} E\left(-\Delta \mathbb{1}_{\{\Delta<0\}} \Delta (g(W) f(W) - g(W-\Delta) f(W-\Delta))\right) \\
&\leq \frac{1}{2\lambda} E\left(\Delta^* \mathbb{1}_{\{\Delta<0\}} \Delta \left(g(W) f(W) - g\left(W'\right) f\left(W'\right)\right)\right) \\
&= \frac{1}{2\lambda} E\left(\Delta^* \Delta \left(\mathbb{1}_{\{\Delta<0\}} + \mathbb{1}_{\{\Delta>0\}}\right) g(W) f(W)\right) \qquad (17) \\
&= \frac{1}{2\lambda} E\left(\Delta \Delta^* g(W) f(W)\right) \\
&\leq \frac{1}{2\lambda} E\left|E\left(\Delta \Delta^* \mid W\right)\right|
\end{aligned}$$

$E\left(\Delta^* \Delta \mathbb{1}_{\{\Delta<0\}} g\left(W'\right) f\left(W'\right)\right) = -E\left(\Delta^* \Delta 1_{\{\Delta>0\}} g(W) f(W)\right)$ because of the exchangeability of W and W', and $|g(w) f(w)| \leq 1$ for all $w \in \mathbb{R}$.

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Similarly we have

$$\frac{1}{2\lambda}\mathrm{E}(\Delta\mathbb{1}_{\{\Delta>0\}}\Delta\left(\mathbb{1}_{\{W-\Delta\le z\}}-\mathbb{1}_{\{W\le z\}}\right)\le\frac{1}{2\lambda}\mathrm{E}\left|\mathrm{E}\left(\Delta\Delta^{*}\mid W\right)\right|. \tag{18}$$

Combining (16), (17) and (18)

$$I_1\le\frac{1}{\lambda}\mathrm{E}\left|\mathrm{E}\left(\Delta\Delta^{*}\mid W\right)\right| \tag{19}$$

Following the same argument, we also have

$$I_1\ge-\frac{1}{\lambda}\mathrm{E}\left|\mathrm{E}\left(\Delta\Delta^{*}\mid W\right)\right| \tag{20}$$

proved □

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Normal Approximation

Normal approximation is a special case of the nonnormal with the difference for the Stein's solution can be bounded by 1.

Theorem 2.5 (Normal Approximation)

Let (W, W') be an exchangeable pair satisfying

$$\mathrm{E}(\Delta \mid W) = \lambda(W + R) \tag{21}$$

for some constant $\lambda \in (0,1)$ and random variable R, where $\Delta = W - W'$. Then

$$\begin{aligned}&\sup_{z\in\mathbb{R}} |\mathrm{P}(W \le z) - \Phi(z)| \\ &\quad \le \mathrm{E}\left|1 - \frac{1}{2\lambda}\mathrm{E}\left(\Delta^2 \mid W\right)\right| + \mathrm{E}|R| + \frac{1}{\lambda}\mathrm{E}\left|\mathrm{E}\left(\Delta\Delta^* \mid W\right)\right|\end{aligned}$$

where $\Delta^* := \Delta^*(W, W')$ is any random variable satisfying $\Delta^*(W', W) = \Delta^*(W, W')$ and $\Delta^* \ge |\Delta|$.

Normal Approximation

Corollary 2.7

If $|\Delta| \leq \delta$ and $\mathrm{E}|W| \leq 2$, then

$$\sup_{z\in\mathbb{R}} |\mathrm{P}(W \leq z) - \Phi(z)| \leq \mathrm{E}\left|1 - \frac{1}{2\lambda}\mathrm{E}\left(\Delta^2 \mid W\right)\right| + \mathrm{E}|R| + 3\delta.$$

Proof. When $\mathrm{E}|R| \geq 1$, LHS ≤ 1 holds. It remains to consider $\mathrm{E}|R| < 1$. Let $\Delta^* = \delta \geq |\Delta|$. Then,

$$\begin{aligned}
\frac{1}{\lambda}\mathrm{E}|\mathrm{E}(\Delta\Delta^* \mid w)| &= \frac{1}{\lambda}\mathrm{E}|\delta\mathrm{E}(\Delta)| \\
&= \frac{\delta}{\lambda}\mathrm{E}|\lambda(w + R)| = \delta\mathrm{E}|w + R| \\
&\leq \delta(\mathrm{E}|w| + \mathrm{E}|R|) \\
&\leq 2\delta + \delta\mathrm{E}|R| \\
&< 3\delta
\end{aligned}$$

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In comparison to

$$\sup_{z\in\mathbb{R}} |P(W \le z) - \Phi(z)| \le \mathrm{E}\left|1 - \frac{1}{2\lambda}\mathrm{E}\left(\Delta^2 \mid W\right)\right| + \mathrm{E}|R| + 1.5\delta + \delta^3/\lambda \tag{22}$$

by Rinott and Rotar. When $|\Delta| \le \delta$, and assuming $\mathrm{E}|W| \le 2$, Corollary 2.7 is an improvement of (22)

$$\min\left(1, \mathrm{E}\left|1 - \frac{1}{2\lambda}\mathrm{E}\left(\Delta^2 \mid W\right)\right| + \delta\right) \le 2\min\left(1, \mathrm{E}\left|1 - \frac{1}{2\lambda}\mathrm{E}\left(\Delta^2 \mid W\right)\right| + \delta^3/\lambda\right)$$

Normal Approximation

It follows from the Cauchy inequality that for any $a > 0$,

$$|\Delta| \leq a/2 + \Delta^2/(2a)$$

Thus, we can choose $\Delta^* = a/2 + \Delta^2/(2a)$ with a proper constant a and obtain the following corollary.

Corollary 2.9

Assume that $\mathrm{E}|W| \leq 2$. Then, under the condition of Theorem 2.1,

$$\sup_{z\in\mathbb{R}} |\mathrm{P}(W \leq z) - \Phi(z)| \leq \mathrm{E}\left|1 - \frac{1}{2\lambda}\mathrm{E}\left(\Delta^2 \mid W\right)\right| + \mathrm{E}|R| + 2\sqrt{\frac{\mathrm{E}\left|\mathrm{E}\left(\Delta^3 \mid W\right)\right|}{\lambda}}$$

Proof. Similarly consider when $\mathrm{E}|R| < 1$, $\Delta^* = \frac{a}{2} + \frac{\Delta^2}{2a}$

$$\begin{aligned}
\frac{1}{\lambda} E\left|E\left(\Delta \Delta^* \mid w\right)\right| &\leqslant \frac{1}{\lambda}\frac{a}{2} E|(\Delta \mid w)| + \frac{1}{\lambda}\frac{1}{2a} E\left|E\left(\Delta^3 \mid w\right)\right| \\
&= \frac{a}{2} E|w+R| + \frac{1}{\lambda}\frac{1}{2a} E\left|E\left(\Delta^3 \mid w\right)\right| \\
&\leqslant 2a + \frac{1}{2a}\frac{1}{\lambda} E\left|E\left(\Delta^3 \mid w\right)\right| \\
&\leqslant 2\sqrt{\frac{E \mid E\left(\Delta^3 \mid w\right) \mid}{\lambda}}
\end{aligned}$$

In comparison to

$$\begin{aligned}
\sup_{z \in \mathbb{R}} |P(W \leq z) - \Phi(z)| \leq \mathrm{E}\left|1 - \frac{1}{2\lambda}\mathrm{E}\left(\Delta^2 \mid W\right)\right| \\
+ \mathrm{E}|R| + \left(\frac{\mathrm{E}|\Delta|^3}{\lambda}\right)^{1/2}
\end{aligned} \tag{23}$$

Clearly, $\mathrm{E}\left|\mathrm{E}\left(\Delta^3 \mid W\right)\right| \leq \mathrm{E}|\Delta|^3$, Corollary 2.9 improves (23).

bounded Case

For any absolutely continuous function f for which the expectations below exist, recalling $\Delta = W - W'$, by exchangeability we have (see (3))

$$\begin{aligned} 0 &= E\left(W - W'\right)\left(f\left(W'\right) + f(W)\right) \\ &= 2\lambda(Ef(W)g(W) + Ef(W)R - E\int_{-\infty}^{\infty} f'(W+t)\hat{K}(t)dt) \end{aligned} \tag{24}$$

$$\hat{K}(t) = \frac{1}{2\lambda}E\{\Delta(\mathbf{1}\{-\Delta \le t \le 0\} - 1\{0 < t \le -\Delta\}) \mid W\} \tag{25}$$

Note that here, we have

$$\int_{-\infty}^{\infty} \hat{K}(t)dt = \frac{1}{2\lambda}E\left(\Delta^2 \mid W\right) \tag{26}$$

$$Ef(W)g(W) + Ef(W)R(W) = E\int_{-\infty}^{\infty} f'(W+t)\hat{K}(t)dt \tag{27}$$

For a given function $g(y)$, let Y be a random variable with density function $p(y)$,

$$p(y) = c_1 e^{-G(y)}, \text{ where } G(y) = \int_0^y g(t)dt \tag{28}$$

with

$$c_1^{-1} = \int_{-\infty}^{\infty} e^{-G(y)} dy < \infty \tag{29}$$

Note that (28) implies

$$p'(y) = -g(y)p(y) \quad \text{for all } y \in (a, b). \tag{30}$$

Lemma 2.11

Let p be a density function and let

$$F(y) = \int_{-\infty}^{y} p(x)dx$$

be the associated distribution function. Further, let h be a measurable function and f_h the Stein solution. Suppose there exist $d_1 > 0$ and $d_2 > 0$ such that for all y we have

$$\min(1 - F(y), F(y)) \leq d_1 p(y)$$

$$\left|p'(y)\right| \min(F(y), 1 - F(y)) \leq d_2 p^2(y).$$

Then if h is bounded

$$\|f_h\| \leq 2d_1\|h\| \tag{31}$$

$$\left\|f_h p'/p\right\| \leq 2d_2\|h\| \tag{32}$$

$$\|f_h'\| \leq (2 + 2d_2)\|h\|. \tag{33}$$

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For some cases, the following two conditions will help verify the hypotheses of Lemma 2.11 for densties of the form (28).

Conditions:

- (H1) The function $g(y)$ is non-decreasing and $yg(y) \geq 0$;
- (H2) The function g is abeolutely continuous, and there exists $c_2 < \infty$ such that

$$\min\left(\frac{1}{c_1}, \frac{1}{|g(y)|}\right)\left(|y| + \frac{3}{c_1}\right)\max\left(1, |g'(y)|\right) \leq c_2$$

Lemma 2.12

Suppose that the density p is given by (28), and g satisfying Conditions (H1) (H2). and $E|g(Y)| < \infty$ for Y having density p. Then conditions and all the bounds in Lemma 2.11 on the solution f and its derivatives hold, with $d_1 = 1/c_1, d_2 = 1$ for all y.

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Theorem 2.13

Let (W, W') be an exchangeable pair satisfying (3), and let Y have density (28), and g in (3) satisfying $E|g(Y)| < \infty$ and Conditions (H1)and (H2). If $\Delta = W - W'$ satisfies $|W - W'| \leq \delta$ for some constant δ then

$$\begin{aligned} &\sup_{z\in\mathbb{R}} |P(W \leq z) - P(Y \leq z)| \\ \leq \quad & 3E \left| 1 - \frac{1}{2\lambda} E\left(\Delta^2 \mid W\right)\right| + c_1 \max\{1, c_2\}\delta + \frac{2}{c_1} E|R| \\ & + \delta^3 \lambda^{-1} \left\{\left(2 + \frac{c_2}{2}\right) E|g(W)| + \frac{c_1 c_2}{2}\right\} \end{aligned} \tag{34}$$

proof: Since (34) is trivial when $c_1 c_2 \delta > 1$, we assume

$$c_1 c_2 \delta \leq 1 \tag{35}$$

Let F be the distribution function of Y and for $z \in \mathbb{R}$ let $f = f_z$ be the solution to the Stein equation

$$f'(w) - f(w)g(w) = \mathbf{1}(w \leq z) - F(z).$$

By Lemma 2.12, the bound (31) of Lemma 2.11 holds, so $\|f\| < \infty$. Letting $\hat{K}(t)$ be given by (25), in view of identities (26), (27), and that $|W - W'| \leq \delta$ and $\hat{K}(t) \geq 0$, we obtain

$$\begin{aligned}
&Ef(W)g(W) + Ef(W)r(W) \\
&= E\int_{-\infty}^{\infty} f'(W+t)\hat{K}(t)dt \\
&= E\int_{-\delta}^{\delta}\{f(W+t)g(W+t) + \mathbf{1}(W+t \leq z) - F(z)\}\hat{K}(t)dt \\
&\geq E\int_{-\delta}^{\delta} f(W+t)g(W+t)\hat{K}(t)dt + \tfrac{1}{2\lambda}\left(E\mathbf{1}_{\{W \leq z-\delta\}}\Delta^2 - F(z)E\Delta^2\right)
\end{aligned} \tag{36}$$

Using $F'(z) = p(z) \leq c_1$ by (28), we have

$$
\begin{aligned}
&\frac{1}{2\lambda}\left(E\mathbf{1}_{\{W\leq z-\delta\}}\Delta^2 - F(z)E\Delta^2\right) \\
&\quad = \left(E\mathbf{1}_{\{W\leq z-\delta\}} - F(z-\delta)\right) \\
&\qquad - E\left\{\left(\mathbf{1}_{\{W\leq z-\delta\}} - F(z)\right)\left(1 - \frac{1}{2\lambda}E\left(\Delta^2 \mid W\right)\right)\right\} \\
&\qquad + \left(F(z-\delta) - F(z)\right) \\
&\quad \geq \left(P(W \leq z-\delta) - F(z-\delta)\right) \\
&\qquad - E\left| 1 - \frac{1}{2\lambda}E\left(\Delta^2 \mid W\right)\right| - c_1\delta
\end{aligned} \tag{37}
$$

Combining(36), (37), we have

$$\begin{aligned} & P(W \leq z-\delta)-F(z-\delta) \\ \leq & Ef(W)g(W)+Ef(W)R-E\int_{-\delta}^{\delta} f(W+t)g(W+t)\hat{K}(t)dt \\ & +E \mid 1-\frac{1}{2\lambda}E\left(\Delta^2 \mid W\right) \mid +c_1\delta \end{aligned} \tag{38}$$

As $c_1\delta \leq c_1 \max\left(1, c_2\right)\delta$,if we want to prove one side of (34),prove the following is enough.

$$\begin{aligned} & Ef(W)g(W)+Ef(W)R-E\int_{-\delta}^{\delta} f(W+t)g(W+t)\hat{K}(t)dt \\ \leq & \quad 2E\left|1-\frac{1}{2\lambda}E\left(\Delta^2 \mid W\right)\right|+\frac{2}{c_1}E|r(W)| \\ & +\delta^3\lambda^{-1}\left\{\left(2+c_2/2\right)E|g(W)|+c_1c_2/2\right\} \end{aligned} \tag{39}$$

$$
\begin{aligned}
& Ef(W)g(W) + Ef(W)R - E\int_{-\delta}^{\delta} f(W+t)g(W+t)\hat{K}(t)dt \\
&= Ef(W)g(W)\left(1 - \frac{1}{2\lambda}E\left(\Delta^2 \mid W\right)\right) + Ef(W)R \\
&+ E\int_{-\delta}^{\delta}\{f(W)g(W) - f(W+t)g(W+t)\}\hat{K}(t)dt \\
&:= J_1 + J_2 + J_3.
\end{aligned} \tag{40}
$$

Lemma 2.12 and (31), (32), and (33) of Lemma 2.11 yield, along with (28), that

$$\|f\| \leq 2/c_1, \quad \|fg\| \leq 2 \quad and \quad \left\|f'\right\| \leq 4. \tag{41}$$

$$|J_1| \leq 2E\left|1 - \frac{1}{2\lambda}E\left(\Delta^2 \mid W\right)\right| \tag{42}$$

$$|J_2| \leq (2/c_1)\, E|R| \tag{43}$$

To bound J_3, we first show that

$$\sup_{|t|\le\delta} |g(w+t)-g(w)| \le \frac{c_1 c_2 \delta}{2}\left(c_1+|g(w)|\right). \tag{44}$$

From Condition (H2) it follows that

$$\begin{aligned} |g'(w)| &\le \frac{c_1 c_2}{3\min\left(1/c_1, 1/|g(w)|\right)} \\ &= \frac{c_1 c_2}{3}\max\left(c_1, |g(w)|\right) \\ &\le \frac{c_1 c_2}{3}\left(c_1+|g(w)|\right) \end{aligned} \tag{45}$$

Thus by the mean value theorem

$$\sup_{|t|\le\delta}|g(w+t)-g(w)|$$

$$\le \delta\sup_{|t|\le\delta}|g'(w+t)| \tag{46}$$

$$\le \frac{c_1c_2\delta}{3}(c_1+|g(w)|)+\frac{1}{3}\sup_{|t|\le\delta}|g(w+t)-g(w)|$$

by (35). This proves (44). Now, by (41) and (44), when $|t|\le\delta$,

$$\begin{aligned}
&|f(w)g(w)-f(w+t)g(w+t)|\\
&\quad\le |g(w)||f(w+t)-f(w)|+|f(w+t)||g(w+t)-g(w)|\\
&\le 4|g(w)||t|+\frac{2}{c_1}\frac{c_1c_2\delta}{2}(c_1+|g(w)|)\\
&\le (4+c_2)\,\delta|g(w)|+\delta c_1c_2
\end{aligned} \tag{47}$$

Therefore

$$|J_3| \leq (4+c_2)\,\delta E\left(|g(W)|\Delta^2/(2\lambda) + \delta c_1 c_2 E\Delta^2/(2\lambda)\right. \tag{48}$$

$$\leq \delta^3\lambda^{-1}\left\{(2+c_2/2)\,E|g(W)| + c_1c_2/2\right\}$$

Combining $(40), (42), (43)$ and (47) ,we prove the(39). Similarly, one can demonstrate

$$\begin{array}{l} F(z+\delta) - P(W \leq z+\delta) \\ \leq \quad 3E \mid 1 - \frac{1}{2\lambda} E\left(\Delta^2 \mid W\right)\right) \mid + c_1\delta + 2E \mid R \mid /c_1 \\ \quad +\delta^3\lambda^{-1}\left\{(2+c_2/2)\,E|g(W)| + c_1c_2/2\right\} \end{array}$$

As $c_1\delta \leq c_1 \max(1, c_2)\,\delta$, the proof of (34) is complete.

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Quadratic Forms

Theorem 3.1

Let $X_1, X_2, \ldots$ be i.i.d. random variables with a zero mean, unit variance and a finite fourth moment. Let $A = (a_{ij})_{i,j=1}^n$ be a real symmetric matrix with $a_{ii} = 0$ for all $1 \le i \le n$ and $\sigma_n^2 = 2\sum_{i=1}^n \sum_{j=1}^n a_{ij}^2 . Put\ W_n = \frac{1}{\sigma_n}\sum_{i\neq j} a_{ij}X_iX_j$. Then

$$\begin{aligned}&\sup_{x\in\mathbb{R}} \left|\mathrm{P}\left(W_n \le x\right) - \Phi(x)\right| \\ &\le \frac{C\mathrm{E}X_1^4}{\sigma_n^2}\left(\sqrt{\sum_i \left(\sum_j a_{ij}^2\right)^2} + \sqrt{\sum_{i,j}\left(\sum_k a_{ik}a_{jk}\right)^2}\right)\end{aligned} \tag{49}$$

where C is an absolute constant.

proof: Let $W_n = h(X_1, \ldots, X_n)$, $W_n' = h(X_1, \ldots, X_I', \ldots, X_n)$. where I be a random index uniformly distributed over $\{1, \ldots, n\}$ independent of any other random variable.and

$$X_i' \Big| X_j, j \neq i \quad \overset{\mathrm{d}}{\sim} \quad X_i \Big| X_j, j \neq i$$

Then (W, W') is an exchangeable pair.

$$\Delta = W_n - W_n' = \frac{2}{\sigma_n} \sum_{j \neq I} a_{jI} X_j (X_I - X_I')$$

Let $X = \sigma(X_1, \ldots, X_n)$

$$\begin{aligned} \mathrm{E}(\Delta \mid X) &= \frac{2}{\sigma_n} \sum_{i=1}^{n} \sum_{j \neq i} \mathrm{E}\left(a_{ji} X_j (X_i - X_i') \mid X\right) \\ &= \frac{2}{n} W_n \end{aligned}$$

As such, condition (21) holds with $\lambda = 2/n$ and $R = 0$.

$$\frac{1}{2\lambda}\mathrm{E}\left(\Delta^2 \mid X\right) = \frac{1}{\sigma_n^2}\sum_{i=1}^{n}\left(X_i^2+1\right)\left(\sum_{j=1}^{n} a_{ij}X_j\right)^2$$

Note that by the assumptions $\sigma_n^2 = 2\sum_{i,j} a_{ij}^2$ and $a_{ii} = 0$,thus

$$\mathrm{E}\left(\frac{1}{2\lambda}\mathrm{E}\left(\Delta^2 \mid X\right)\right) = 1$$

$$\begin{aligned} |1-\frac{1}{2\lambda}E\left(\Delta^2 \mid W_n\right)|^2 &= |E\left[E\left(1-\frac{1}{2\lambda}\Delta^2 \mid \chi\right) \mid W_n\right]|^2 \\ &\leq E\left\{\left[E\left(1-\frac{1}{2\lambda}\Delta^2 \mid \chi\right)\right]^2 \mid W_n\right\} \end{aligned}$$

Then

$$\mathrm{E}\left|1-\frac{1}{2\lambda}\mathrm{E}\left(\Delta^2 \mid W_n\right)\right|^2 \leq \mathrm{Var}\left(\frac{1}{\sigma_n^2}\sum_{i=1}^{n}\left(X_i^2+1\right)\left(\sum_{j=1}^{n} a_{ij}X_j\right)^2\right)$$

Observe that

$$\begin{aligned}&\mathrm{Var}\left(\sum_{i=1}^n \left(X_i^2+1\right)\left(\sum_{j=1}^n a_{ij}X_j\right)^2\right)\\&=\sum_{i=1}^n \mathrm{Var}\left(\left(X_i^2+1\right)\left(\sum_{j=1}^n a_{ij}X_j\right)^2\right)\\&\quad+\sum_{i\neq i'} \mathrm{Cov}\left(\left(X_i^2+1\right)\left(\sum_{j=1}^n a_{ij}X_j\right)^2, \left(X_{i'}^2+1\right)\left(\sum_{k=1}^n a_{i'k}X_k\right)^2\right)\end{aligned} \tag{50}$$

For the first term, recalling that $a_{ii}=0$ for all $1\leq i\leq n$, we have

$$\begin{aligned}&\sum_{i=1}^n \mathrm{Var}\left(\left(X_i^2+1\right)\left(\sum_{j=1}^n a_{ij}X_j\right)^2\right)\\&\leq \sum_{i=1}^n \mathrm{E}\left(X_i^2+1\right)^2 \mathrm{E}\left(\sum_{j=1}^n a_{ij}X_j\right)^4\\&\leq C\left(\mathrm{E}\left(X_1^4\right)\right)^2\sum_{i=1}^n\left(\sum_{j=1}^n a_{ij}^2\right)^2\end{aligned} \tag{51}$$

where C is an absolute constant.

To bound the second term of (50), for any $i \neq k$ define

$$M_i = (X_i^2+1)\left(\sum_{j=1}^n a_{ij}X_j\right)^2$$
$$M_i^{(k)} = (X_i^2+1)\left(\sum_{j\neq k}^n a_{ij}X_j\right)^2$$

For the second term of (50), for any $i \neq i'$, we have

$$\begin{aligned}
&\mathrm{Cov}\left((X_i^2+1)\left(\sum_{j=1}^n a_{ij}X_j\right)^2, (X_{i'}^2+1)\left(\sum_{k=1}^n a_{i'k}X_k\right)^2\right) \\
&= \mathrm{Cov}\left(M_i, M_{i'}\right) \\
&= \mathrm{Cov}\left(M_i^{(i')}, M_{i'}\right) + \mathrm{Cov}\left(M_i, M_{i'}^{(i)}\right) \\
&\quad - \mathrm{Cov}\left(M_i^{(i')}, M_{i'}^{(i)}\right) + \mathrm{Cov}\left(M_i - M_i^{(i')}, M_{i'} - M_{i'}^{(i)}\right)
\end{aligned} \tag{52}$$

Given $\mathcal{F}_{ii'} := \sigma\{X_j, j \neq i, i'\}$, random variables $M_i^{(i')}$ and $M_{i'}^{(i)}$ are independent.

$$\mathrm{Cov}\left(M_i^{(i')}, M_{i'}^{(i)}\right)$$

$$= \mathrm{Cov}\left(\mathrm{E}\left(\left(X_i^2+1\right)\left(\sum_{j\neq i'}^n a_{ij}X_j\right)^2 \mid \mathcal{F}_{ii'}\right)\right.$$
$$\left.\mathrm{E}\left(\left(X_{i'}^2+1\right)\left(\sum_{k\neq i}^n a_{i'j}X_k\right)^2 \mid \mathcal{F}_{ii'}\right)\right)$$
$$= 4\,\mathrm{Cov}\left(\left(\sum_{j\neq i'}^n a_{ij}X_j\right)^2, \left(\sum_{k\neq i}^n a_{i'k}X_k\right)^2\right)$$
$$\leq C\sum_{j=1}^n a_{ij}^2 a_{i'j}^2 \mathrm{E}\left(X_1^4\right) + C\left(\sum_{k=1}^n a_{ik}a_{i'k}\right)^2$$

Similar arguments hold for other terms of (52). Hence,

$$\sum_{i\neq i'} \mathrm{Cov}\left(\left(X_i^2+1\right)\left(\sum_{j=1}^n a_{ij}X_j\right)^2, \left(X_{i'}^2+1\right)\left(\sum_{k=1}^n a_{i'k}X_k\right)^2\right)$$
$$\leq C\mathrm{E}\left(X_1^4\right)^2\left(\sum_{i=1}^n\left(\sum_{j=1}^n a_{ij}^2\right)^2 + \sum_{1\leq i,j\leq n}\left(\sum_{k=1}^n a_{ik}a_{jk}\right)^2\right) \quad (53)$$

It follows from (50), (51) and (53) that

$$\mathrm{E}\left|1-\frac{1}{2\lambda}\mathrm{E}\left(\Delta^2 \mid W_n\right)\right| \leq C\sigma_n^{-2}\mathrm{E}\left(X_1^4\right)\left(\sqrt{\sum_i\left(\sum_j a_{ij}^2\right)^2}+\sqrt{\sum_{i,j}\left(\sum_k a_{ik}a_{jk}\right)^2}\right) \quad (54)$$

Similarly, we can have:

$$\frac{1}{\lambda}\mathrm{E}\left|\mathrm{E}\left(\Delta|\Delta| \mid W_n\right)\right| \leq C\sigma_n^{-2}\mathrm{E}\left(X_1^4\right)\left(\sqrt{\sum_i\left(\sum_j a_{ij}^2\right)^2}+\sqrt{\sum_{i,j}\left(\sum_k a_{ik}a_{jk}\right)^2}\right) \quad (55)$$

This completes the proof of Theorem 3.1 by (54) and (55).

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